

BLOW-ANALYTIC EQUIVALENCE OF TWO VARIABLE REAL ANALYTIC FUNCTION GERMS

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ABSTRACT. Blow-analytic equivalence is a notion for real analytic function germs, introduced by Tzee-Char Kuo in order to develop real analytic equisingularity theory. In this paper we give complete characterisations of blow-analytic equivalence in the two dimensional case: in terms of the real tree model for the arrangement of real parts of Newton-Puiseux roots and their Puiseux pairs, and in terms of minimal resolutions. These characterisations show that in the two dimensional case the blow-analytic equivalence is a natural analogue of topological equivalence of complex analytic function germs. Moreover, we show that in the two-dimensional case the blow-analytic equivalence can be made cascade, and hence satisfies several geometric properties. It preserves, for instance, the contact orders of real analytic arcs.

In the general n -dimensional case, we show that a singular real modification satisfies the arc-lifting property.

A classical result of Burau [4] and Zariski [34] shows the embedded topological type of a plane curve singularity $(X, 0) \subset (\mathbb{C}^2, 0)$ is determined by the Puiseux pairs of each irreducible component and the intersection numbers of any pairs of distinct components. It can be shown, cf. [30], that the topological type of function germs $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ is completely characterised, also in the non-reduced case $f = \prod f_i^{d_i}$, by the embedded topological type of its zero set and the multiplicities d_i of its irreducible components.

In this paper we give a real analytic counterpart of these results and show that the two variable version of blow-analytic equivalence of Kuo is classified by invariants similar to Puiseux pairs, multiplicities of irreducible components, and intersection numbers. Moreover we show several natural geometric properties of this equivalence, answering previously posed questions. In the main result of this paper we give complete characterisation of blow-analytic equivalence classes of two variable real analytic function germs.

Theorem 0.1. *Let $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ and $g : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ be real analytic function germs. Then the following conditions are equivalent:*

- (1) *f and g are blow-analytically equivalent.*
- (2) *f and g have weakly isomorphic minimal resolution spaces.*
- (3) *The real tree models of f and g are isomorphic.*

Moreover if f and g are blow-analytically equivalent then they are equivalent by a cascade-blow-analytic homeomorphism.

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Theorem 0.1 can be stated in both the oriented and non-oriented case, see section 8 below. By a weak isomorphism of resolution spaces we mean a homeomorphism that preseves the basic numerical data of resolutions, see subsection 1.3. The real tree model is a counterpart of Kuo and Lu's tree model [23], a combinatorial object that encodes the numerical data given by the contact orders between the Newton-Puiseux roots of f in the complex case. Cascade-blow-analytic homeomorphisms satisfy many geometric and analytic properties: they lift to the resolution spaces of f and g , they preserve the intersection numbers between real analytic arcs and their Puiseux exponents.

In the general (n -dimensional) case the notion of blow-analytic equivalence is very technical but we need to recall its definition and main properties.

0.1. Blow-analytic equivalence. In a search for a "right" equivalence relation of real analytic function germs, that could play a similar role to the topological equivalence in the complex analytic set-up, at the end of 1970 Tzee-Char Kuo proposed the notion of blow-analytic equivalence [20, 21, 22, 24, 25, 26, 27, 28]). In [28], Kuo proved that blow-analytic equivalence is an equivalence relation and established the local finiteness of blow-analytic types for analytic families of real analytic function-germs with isolated singularities.

We say that a homeomorphism germ $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ is a *blow-analytic homeomorphism* if there exist real modifications $\mu : (M, \mu^{-1}(0)) \rightarrow (\mathbb{R}^n, 0)$, $\tilde{\mu} : (\tilde{M}, \tilde{\mu}^{-1}(0)) \rightarrow (\mathbb{R}^n, 0)$ and an analytic isomorphism $\Phi : (M, \mu^{-1}(0)) \rightarrow (\tilde{M}, \tilde{\mu}^{-1}(0))$ so that the following diagram is commutative:

$$(0.1) \quad \begin{array}{ccc} (M, \mu^{-1}(0)) & \xrightarrow{\mu} & (\mathbb{R}^n, 0) \\ \Phi \downarrow & & \downarrow h \\ (\tilde{M}, \tilde{\mu}^{-1}(0)) & \xrightarrow{\tilde{\mu}} & (\mathbb{R}^n, 0) \end{array}$$

We say that two real analytic function germs $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ and $g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ are *blow-analytically equivalent* if there exists a blow-analytic homeomorphism $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ such that $f = g \circ h$.

Kuo's definition of real modification is very technical and often difficult to work with. We discuss it in section 2. For $n = 2$, we show the following simple characterisation.

Theorem 0.2. *Let X, S be connected nonsingular real analytic surfaces and let $\sigma : X \rightarrow S$ be a proper surjective real analytic map. Then σ is a real modification in the sense of Kuo if and only if it is a composition of point blowings-up.*

Several results showing that two function germs are blow-analytically equivalent were obtained using, mostly toric, equiresolutions by Kuo, Fukui-Yoshinaga [6], Fukui-Paunescu [10], Abderrahmane [1], and others.

Invariants allowing to distinguish different blow-analytic types were constructed, using the geometry of arc-spaces and motivic integration, by Fukui [7], Fichou [5], and in [18]. These constructions are based on the observation that blow-analytic homeomorphisms send real analytic arcs to real analytic arcs. It follows from Theorem 2.3 that shows that the real modifications satisfy the arc lifting property, compare [7] section 3.

For more on the blow-analytic equivalence see the surveys [8], [11].

0.2. Cascade blow-analytic homeomorphisms and their geometric properties.

In general, a blow-analytic homeomorphism h is not necessarily Lipschitz. In two dimensional case, if h gives blow analytic equivalence between analytic function germs, then h is cascade and satisfies many geometric properties.

We say that $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ is a *cascade blow-analytic homeomorphism* if there exists a commutative diagram

$$(0.2) \quad \begin{array}{ccccccc} (M_k, E_k) & \xrightarrow{b_k} & (M_{k-1}, E_{k-1}) & \xrightarrow{b_{k-1}} & \cdots & \xrightarrow{b_2} & (M_1, E_1) & \xrightarrow{b_1} & (\mathbb{R}^2, 0) \\ \Phi \downarrow & & \downarrow h_{k-1} & & & & \downarrow h_1 & & \downarrow h \\ (\tilde{M}_k, \tilde{E}_k) & \xrightarrow{\tilde{b}_k} & (\tilde{M}_{k-1}, \tilde{E}_{k-1}) & \xrightarrow{\tilde{b}_{k-1}} & \cdots & \xrightarrow{\tilde{b}_2} & (\tilde{M}_1, \tilde{E}_1) & \xrightarrow{\tilde{b}_1} & (\mathbb{R}^2, 0), \end{array}$$

where b_i, \tilde{b}_i are point blowings-up, \tilde{E}_i, E_i are the inverse images of the origin, h_i are homeomorphisms, and Φ is an analytic isomorphism. We say that the real analytic function germs $f(x, y), g(x, y)$ are *cascade blow-analytically equivalent* if there exists a cascade blow-analytic homeomorphism h such that $f = g \circ h$.

Suppose that f and g are blow-analytically equivalent. The key step in showing (1) \Rightarrow (2) of theorem 0.1 is to establish the existence of h_1 in (0.2), or equivalently, that in (0.1), $\Phi(E_1) = \tilde{E}_1$, where here $E_1 \subset M$, resp. $\tilde{E}_1 \subset \tilde{M}$, denotes the strict transform of the exceptional divisor of first point blowing-up in $\mu, \tilde{\mu}$ resp.. This is shown in section 3 using the combinatorial properties of dual graphs of real resolutions. This also shows that blow-analytic equivalence implies the cascade one in two variable case.

As we show in section 5 the cascade blow-analytic homeomorphisms satisfy many important geometric properties. They preserve the Puiseux characteristic sequence of real analytic arcs and, in the oriented case, the signs of coefficients at the Puiseux characteristic exponents. They preserve also the order of contact between such arcs. These properties are crucial for the proof of (1) \iff (3) of Theorem 0.1. The theory of real analytic arcs, and more precisely their demi-branches, is developed in section 4.

Kobayashi and Kuo constructed in [16] examples of blow-analytic homeomorphisms $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ that are not cascade. Their examples do not satisfy $\Phi(E_1) = \tilde{E}_1$ and do not preserve the tangency of curves and may send a smooth arc to a singular one and vice versa. Such blow-analytic homeomorphism cannot give blow-analytic equivalence between two real analytic function germs.

0.3. Real tree model. In [23] Kuo and Lu introduced a tree model $T(f)$ of a complex analytic function germ $f(x, y)$. This model allows one to visualise the numerical data given by the contact orders between the Newton-Puiseux roots of f , in particular their Puiseux characteristic exponents.

For $f(x, y)$ real analytic "the real part of $T(f)$ " was proposed by Kurdyka and Paunescu in [29]. In section 6 we propose a similar, but more precise, construction of a real tree model that determines the resolution process of f . Our real tree model is a combinatorial object that encodes the contact orders between the real parts of complex Newton-Puiseux roots of f . It contains the information about the signs of coefficients at the Puiseux characteristic exponents. Thanks to the geometric properties of cascade blow-analytic homeomorphisms, see Theorem 5.1, the proof of (1) \iff (3) of Theorem 0.1 is based on a

fairly straightforward computation of the tree model of the blown-up singularity in terms of the original tree model.

0.4. Examples. Abderrahmane [2] showed that blow-analytically equivalent weighted homogeneous singular $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ and $g : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ have the same weights. This can be also easily verified by Theorem 0.1, see Example 6.9.

The blow-analytic (non-oriented) classification of Brieskorn two variable singularities $\pm x^p \pm y^q$ was obtained in [18] using the Fukui invariants and the zeta functions. This classification coincides with $(x, y) \rightarrow (\pm x, \pm y)$ classification except for the case p odd and $q = pm$ with m even. In the latter case, additionally, $f(x, y) = x^p - y^q$ and $g(x, y) = x^p + y^q$ are blow analytically equivalent, cf. [18], but they are not analytically equivalent. Moreover, f and g are not bi-lipschitz equivalent. It is shown for $f(x, y) = x^3 - y^6$ and $g(x, y) = x^3 + y^6$ in [12], the proof works in general.

Theorem 0.1 allows us to complete the classification of Brieskorn two variable singularities in the oriented case. It coincides with the non-oriented case except for $f(x, y) = x^p - y^q$ and $g(x, y) = x^p + y^q$, both p and q odd. For these functions f and g are blow-analytically equivalent but not by an orientation preserving homeomorphism, cf. exemple 6.12.

The functions $f(x, y) = x(x^3 - y^5)(x^3 + y^5)$ and $g(x, y) = x(x^3 - y^5)(x^3 - 2y^5)$ have the same Fukui invariants and zeta functions. As follows from Theorem 0.1 they are not blow-analytically equivalent, see Examples 1.1 and 6.6.

0.5. Further developement. Open questions. In [19] we compare various equivalence relations between two variable function germs. Namely we show that two real analytic function germs $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ and $g : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ that are C^1 equivalent have the same real tree models and consequently, by Theorem 0.1, are blow-analytically equivalent. If we assume that f and g only bi-lipschitz equivalent, then though the contact orders between Newton-Puiseux roots and the real parts of complex roots of f and g are preserved, the Puiseux pairs of these roots can be different, and therefore their tree models and blow-analytic types are different. The simplest example is given by $x(x^3 + y^5)$ and $x(x^3 - y^5)$, these two functions are bi-lipschitz but not blow-analytically equivalent by an orientation preserving homeomorphism.

Most of the questions answered in this paper for functions of two real variables remain open in higher dimensions, in particular, the very questions what should be the right precise definitions of the blow-analytic equivalence and of the real modification. Our corollary 2.5 shows that Kuo's blow-analytic homeomorphisms preserve real analytic arcs in the n dimensional case. Under what assumption does the blow-analytic equivalence preserve the contact order between real analytic arcs? Does it satisfy metric properties stated in Corollary 5.3 and Propostion 5.4? Note that these properties would allow one to construct more blow-analytic invariants using the methods of motivic integration, as in [5].

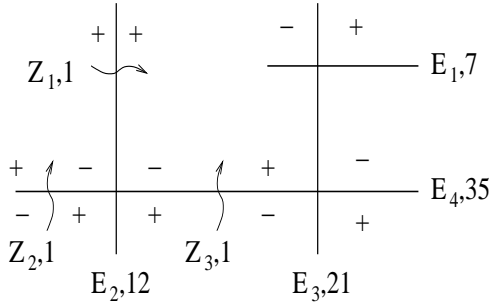
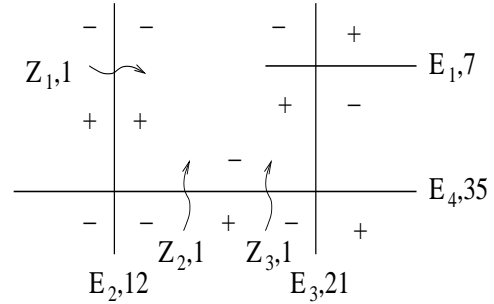
In Lemma 3.4 we show that a function blow-analytically equivalent to normal crossing is itself normalcrossing with the same exponents. Again, we show this result in two variable case. Is it true in the general case?

1. PRELIMINARIES

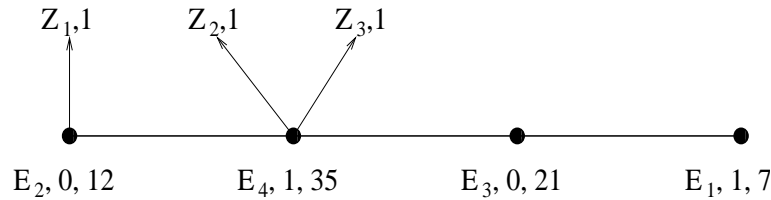
1.1. Dual resolution graph. Let $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ be an analytic function germ. We call a composition of point blowings-up $\mu : M \rightarrow \mathbb{R}^2$ a *resolution of f* if $f \circ \mu$ is normal crossings. Since all blowings-up have point centres there exists a unique minimal resolution of f obtained by blowing-up only the points where the total transform of f is not normal crossings.

Consider the weighted oriented *dual graph* $S = S_{\mu, f}$ associated to a resolution μ of f . Each component E of the exceptional divisor $\mu^{-1}(0)$ corresponds to a vertex of S . For simplicity we denote this vertex also by E . If two such components intersect at a point then the corresponding vertices are joined by an edge. Each component C of the strict transform of $f^{-1}(0)$ is visualized by an arrow drawn at the vertex corresponding to the component E of $\mu^{-1}(0)$ that C intersects. To each vertex E we assign its parity $p_\mu(E)$ and its multiplicity $m(E)$. The multiplicity of the vertex is the generic multiplicity of $f \circ \mu$ on E . The parity of E is 0 if it has an orientable neighbourhood in M . If a tubular neighbourhood of E is a Möbius band then its parity is 1.

Example 1.1. Let $f(x, y) = x(x^3 - y^5)(x^3 + y^5)$ and $g(x, y) = x(x^3 - y^5)(x^3 - 2y^5)$. The resolution graphs of f and g are the following.

Resolution graph of f Resolution graph of g

Note that f and g are desingularised by the same composition of four point blowings-up μ . We denote by E_i 's the components of exceptional divisor of μ , and by Z_j 's the components of the strict transforms of $f^{-1}(0)$ and $g^{-1}(0)$ by μ . On the graph the numbers next to these components denote their multiplicities. The dual graphs of minimal resolution of f and g coincide.



The resolution graphs of f and g give the same Fukui invariants and zeta functions, cf. Theorem I & VII in [15] and formulae (1.1) & (1.2) in [18]. Therefore f and g have the

same Fukui invariants and zeta functions. As follows from Theorem 0.1, see also Example 6.6 below, f and g are not blow-analytically equivalent.

1.2. Invariants of blow-analytic equivalence. We introduce a refinement of Fukui invariant that we will need later. First we recall briefly the construction Fukui invariant, cf. [7]. Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be an analytic function germ. Set

$$A(f) := \{\text{ord}(f(\gamma(t))) \in \mathbb{N} \cup \{\infty\}; \gamma : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^n, 0) \text{ real analytic}\}.$$

Let $\lambda : U \rightarrow \mathbb{R}^n$ be an analytic arc with $\lambda(0) = 0$, where U denotes a neighbourhood of $0 \in \mathbb{R}$. We call λ *nonnegative* (resp. *nonpositive*) for f if $(f \circ \lambda)(t) \geq 0$ (resp. ≤ 0) in a positive half neighbourhood $[0, \delta) \subset U$. Then we set

$$\begin{aligned} A_+(f) &:= \{\text{ord}(f \circ \lambda); \lambda \text{ is a nonnegative arc through } 0 \text{ for } f\}, \\ A_-(f) &:= \{\text{ord}(f \circ \lambda); \lambda \text{ is a nonpositive arc through } 0 \text{ for } f\}. \end{aligned}$$

Fukui in [7] proved that if analytic functions $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ are blow-analytically equivalent, then $A(f) = A(g)$, $A_+(f) = A_+(g)$ and $A_-(f) = A_-(g)$. We call $A(f)$, $A_\pm(f)$ *the Fukui invariant*, *the Fukui invariants with sign*, respectively. The proof of Fukui was based on the fact that a blow-analytic homeomorphism sends a real analytic arc to a real analytic arc, that he proved only for these blow-analytic homeomorphism h defined by the diagram 0.1 with μ and $\tilde{\mu}$ blowings-up of coherent ideals. We complete the general case in Corollary 2.5 below. Apart from the Fukui invariants, motivic type invariants are introduced in [18] and [5].

Let $C_+(f)$ (resp. $C_-(f)$) be the set of local connected components of $\{x \in \mathbb{R}^n; f(x) > 0\}$ (resp. $\{x \in \mathbb{R}^n; f(x) < 0\}$) as set-germs at $0 \in \mathbb{R}^n$. Let $C_+(f) = \{V_1(f), \dots, V_v(f)\}$ and $C_-(f) = \{W_1(f), \dots, W_w(f)\}$. For $i = 1, \dots, v$ (resp. $j = 1, \dots, w$), we call λ a *positive* (resp. *negative*) *arc in* $V_i(f)$ (resp. $W_j(f)$) for f if $(f \circ \lambda)(t) > 0$ (resp. $(f \circ \lambda)(t) < 0$) and $\lambda(t) \in V_i(f)$ (resp. $\lambda(t) \in W_j(f)$) over an open interval $(0, \delta) \subset U$. Then we set

$$\begin{aligned} A_+^i(f) &:= \{\text{ord}(f \circ \lambda); \lambda \text{ is a positive arc in } V_i(f) \text{ for } f\}, \\ A_-^j(f) &:= \{\text{ord}(f \circ \lambda); \lambda \text{ is a negative arc in } W_j(f) \text{ for } f\}, \end{aligned}$$

for $i = 1, \dots, v$, $j = 1, \dots, w$. Using the argument of Fukui [7] and Theorem 2.3 (c) below, the collection of sets $A_+^i(f)$'s and $A_-^j(f)$'s is a blow-analytic invariant. We call them *the refined Fukui invariants with sign*. *The refined Fukui invariant* (without sign) is defined similarly.

1.3. Constructing blow-analytic equivalence.

Definition 1.2. We say that two real analytic functions $f(x, y)$ and $g(x, y)$ have *weakly isomorphic resolution spaces* if there exist resolutions $\mu : M \rightarrow \mathbb{R}^2$, $\tilde{\mu} : \tilde{M} \rightarrow \mathbb{R}^2$ of $f(x, y)$ and $g(x, y)$, respectively, and an analytic isomorphism $\Phi : M \rightarrow \tilde{M}$ such that:

- (1) Φ conserves the exceptional sets and the strict transform of the zero sets:

$$\Phi(\mu^{-1}(0)) = \tilde{\mu}^{-1}(0), \quad \Phi((f \circ \mu)^{-1}(0)) = (g \circ \tilde{\mu})^{-1}(0)$$

- (2) Φ conserves the multiplicities: If C is a component of $(f \circ \mu)^{-1}(0)$ then $\text{mult}_C f \circ \mu = \text{mult}_{\Phi(C)} g \circ \tilde{\mu}$.

(3) Φ conserves the signs: $f \circ \mu(p) > 0$ iff $g \circ \tilde{\mu}(\Phi(p)) > 0$.

The following result follows from Proposition 7.2 of [7].

Proposition 1.3. *If $f(x, y)$ and $g(x, y)$ have weakly isomorphic resolution spaces then f and g are blow-analytically equivalent.*

Remark 1.4. In Definition 1.2 it is enough to assume that Φ is C^∞ or even that it is only a homeomorphism. Indeed, a homeomorphism Φ satisfying (1)-(3) of Definition 1.2 can be approximated by an analytic isomorphism with the same properties. This is fairly easy to see since we are in the two variable case. We can also use the following much deeper Nash approximation argument.

Recall that Nash maps are real analytic maps with semi-algebraic graphs. Firstly, by a result of Shiota [31] we may suppose that f and g are polynomial functions and hence that M and \tilde{M} are Nash manifolds, and that the components of the exceptional divisors and the strict transforms of $f^{-1}(0)$, $g^{-1}(0)$ are their Nash submanifolds resp., intersecting transversally. Then the existence of Φ that is a Nash isomorphism and satisfies (1)-(3) of Definition 1.2 follows directly from the proof of the Nash Isotopy Lemma of [9].

The above argument and Theorem 0.1 show that polynomial function germs $f(x, y)$, $g(x, y)$ (and more generally Nash function germs), are blow-analytically equivalent if and only if they are blow-Nash equivalent, see also [5], [11].

2. REAL MODIFICATIONS

We recall this classical definition of real modification, cf. [28], [8]. We also introduce a slightly more general notion of singular real modification in order to have a notion that is stable by taking the strict transforms by blowings-up with smooth centre, see (b) of Theorem 2.3.

Definition 2.1. Let Y be a real analytic manifold of pure dimension n . We say that $\sigma : X \rightarrow Y$ is a *singular real modification* if the following property is satisfied.

X is a real analytic space, $\sigma : X \rightarrow Y$ is a proper surjective real analytic map, and there exist complexifications $X_{\mathbb{C}}, Y_{\mathbb{C}}$ of X and Y , respectively, and a holomorphic extension $\sigma_{\mathbb{C}} : X_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$ of σ , that satisfy:

$X_{\mathbb{C}}$ is a complex analytic space of pure complex dimension n and $\sigma_{\mathbb{C}}$ is an isomorphism in the complement of a closed nowhere dense subset B of $X_{\mathbb{C}}$. (that is $\sigma_{\mathbb{C}}$ restricted to $X_{\mathbb{C}} \setminus B$ is open and an isomorphism onto its image.)

If, moreover, X is nonsingular, then we say that σ is a *real modification*.

Note that a real analytic map that is an isomorphism in the complement of a closed nowhere dense subset of X is not necessarily a real modification. This is for instance the case for $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ given by $\sigma(x) = x^3$.

Remark 2.2. If one defined real modifications simply as compositions of blowings-up with smooth centres, then one would need the strong factorisation in order to show that the induced notion of blow-analytic equivalence, see subsection 0.1, is an equivalence relation. At the moment we do not know whether the strong factorisation holds. With Kuo's

definition, certainly quite complicated and less natural, the proof of transitivity of the blow-analytic equivalence is fairly easy.

Theorem 2.3. *Let $\sigma : X \rightarrow Y$ be a singular real modification. Then*

- (a) *σ is an isomorphism over the complement in Y of a subanalytic subset $A \subset Y$ of real codimension 2.*
- (b) *Let $\pi : Y' \rightarrow Y$ be a blowing-up with a nonsingular nowhere dense centre. Then the strict transform $\sigma' : X' \rightarrow Y'$ by π is a singular real modification.*
- (c) *If $\gamma : (\mathbb{R}, 0) \rightarrow (Y, p)$ is the germ of a real analytic arc at $p \in Y$ then there is a real analytic $\tilde{\gamma} : (\mathbb{R}, 0) \rightarrow (X, \tilde{p})$, $\tilde{p} \in X$, such that $\sigma \circ \tilde{\gamma} = \gamma$.
Moreover, there is a closed subanalytic nowhere dense $A \subset Y$ (independent of γ) such that if $\gamma^{-1}(A)$ is discrete then such $\tilde{\gamma}$ is unique.
(We say for short that σ satisfies the arc lifting property and the unique generic arc lifting property).*

Proof. Let $\sigma : X \rightarrow Y$ be a singular real modification and let B be a closed nowhere dense subset of $X_{\mathbb{C}}$ such that $\sigma_{\mathbb{C}}$ is an isomorphism in the complement of B .

First we show (b). Let $\pi : Y' \rightarrow Y$ be a blowing-up with smooth nowhere dense centre C and consider the diagram

$$(2.1) \quad \begin{array}{ccc} X & \xleftarrow{\pi'} & X' \\ \sigma \downarrow & & \downarrow \sigma' \\ Y & \xleftarrow{\pi} & Y' \end{array}$$

where $\sigma' : X' \rightarrow Y'$ is the strict transform of σ by π . Then π' is the blowing-up of the pullback by σ of the ideal of C . The above diagram induces the complexified diagram, where we complexify π and π' by the corresponding complex blowings-up. Let $E'_{\mathbb{C}}$ denote the exceptional divisor of $\pi'_{\mathbb{C}}$. Then $\sigma'_{\mathbb{C}}$ is an isomorphism in the complement of $E'_{\mathbb{C}} \cup \pi'^{-1}_{\mathbb{C}}(B)$ that is nowhere dense in $X'_{\mathbb{C}}$.

To show (a) we note first that by assumption the generic fibres of σ consist of single points. Over a generic point of $y \in Y$ in codimension 1, $\sigma_{\mathbb{C}}$ is finite. Indeed, let $y = \sigma(x)$ and suppose that $(\sigma_{\mathbb{C}})_x$ were not finite. Then $X_{\mathbb{C}}$ would contain a vertical component $(X_{\mathbb{C}})_1$ (i.e. on such component the rank of the differential of $\sigma_{\mathbb{C}}$ restricted to the regular part of $(X_{\mathbb{C}})_1$ is everywhere smaller than n). But this contradicts the existence of a nowhere dense B of the definition of real modification. Therefore over y , $\sigma_{\mathbb{C}}$ is finite. But a finite real modification has to be of degree one, that is an isomorphism.

If σ is a blowing-up with a smooth nowhere dense centre, or a composition of such blowings-up, then it satisfies (c). Thus (c) for an arbitrary real modification follows from the local flattening theorem, cf. [13], [14]. Indeed, for any germ $\gamma : (\mathbb{R}, 0) \rightarrow (Y, p)$ there is a composition of local blowings-up with smooth centres $\pi : Y' \rightarrow Y$ such that γ lifts to Y' and that the strict transform $\sigma' : X' \rightarrow X$ of σ by π is flat. Since σ' is a singular real modification by (b) it has to be an isomorphism. Therefore γ lifts to X' and hence to X , see (2.1). The last claim of (c) follows now easily from (a). \square

Remark 2.4. The arc-lifting property for real modifications has been proven also in [11], section 5.

Corollary 2.5. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a blow-analytic homeomorphism and let $\gamma : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^n, 0)$. Then γ is real analytic if and only if so is $h \circ \gamma$.*

Proof of Theorem 0.2. Clearly a composition of point blowings-up is a real modification. We shall show the converse. Suppose that σ is a real modification. By (a) of Theorem 2.3 there exists a discrete subset $A \subset S$ such that σ is an isomorphism over $S \setminus A$. Fix $p \in A$ and suppose that σ is not an isomorphism over p . Let $\pi : \hat{S} \rightarrow S$ be a blowing-up of S at p . The main point is to show that σ factors through π , that is that there is $\hat{\sigma} : X \rightarrow \hat{S}$ such that $\pi \circ \hat{\sigma} = \sigma$. The proof follows a classical argument of elimination of indeterminacy of rational maps between algebraic surfaces, see e.g. [3] Proposition II.8. Since the problem is local, we shall work in a neighbourhood of p and assume that σ is an isomorphism over $S \setminus \{p\}$.

Lemma 2.6. *Let $q \in \sigma^{-1}(p)$ and assume that σ is not an isomorphism at q . Let x, y be a system of local analytic coordinates x, y at p . Then there is a linear combination $z = ax + by \in m_p \setminus m_p^2$ such that $\sigma^*(z) \in m_q^2$.*

Proof. Indeed, by assumption, σ is not an isomorphism at q so $\sigma^*(x)$ and $\sigma^*(y)$ are linearly dependent in m_q/m_q^2 . \square

Let $\hat{\pi} : \hat{X} \rightarrow X$ be the blowing-up of the ideal $\sigma^*(m_p)$. Then \hat{X} can be identified with this irreducible component of $X \times_S \hat{S}$ that is not entirely included in the inverse image of the exceptional divisor E of π .

$$\begin{array}{ccc} X & \xleftarrow{\hat{\pi}} & \hat{X} \subset X \times_S \hat{S} \\ \sigma \downarrow & & \downarrow \hat{\sigma} \\ S & \xleftarrow{\pi} & \hat{S} \end{array}$$

Note that $\hat{\pi} : \hat{X} \rightarrow X$ is surjective since the zero set of $\sigma^*(m_p)$ is nowhere dense in X . By (a) of Theorem 2.3, $\hat{\sigma}$ is an isomorphism over the complement of a finite subset F of the exceptional divisor $E \subset \hat{S}$ and hence the map $\hat{\sigma}^{-1} : \hat{S} \setminus F \rightarrow \hat{X}$ is well-defined.

Lemma 2.7. *$\hat{\pi} \circ \hat{\sigma}^{-1} : E \setminus F \rightarrow \sigma^{-1}(p)$ is not constant.*

Proof. Suppose that this map is constant and that its image is $q \in \sigma^{-1}(p)$. For any $\hat{p} \in E \setminus F$ and for $z \in m_p$ given by lemma 2.6

$$(\hat{\pi} \circ \hat{\sigma}^{-1})^* \sigma^*(z) \in m_{\hat{p}}^2.$$

But this is impossible since

$$(\hat{\pi} \circ \hat{\sigma}^{-1})^* \sigma^*(z) = \pi^*(z) \notin m_{\hat{p}}^2$$

for all $\hat{p} \in E$ but one (the one corresponding to the zero set of z). \square

Thus for any $q \in \sigma^{-1}(p)$, $\hat{\pi}^{-1}(q)$ is finite. It follows from the next lemma that $\hat{\pi}$ has to be an analytic isomorphism.

Lemma 2.8. *Let X be a nonsingular real analytic surface and let $\Pi : \mathcal{X} \rightarrow X$ be the blowing-up of an ideal \mathcal{I} (not identically equal to zero). Suppose that locally at any point of X , \mathcal{I} can be generated by two real analytic functions. Then, if all the fibres of Π are finite then Π is an analytic isomorphism.*

Proof. We work locally on X so we assume that Π is an isomorphism over the complement of a single point $q \in X$. Since \mathcal{I}_q has two generators, $\mathcal{X} \subset X \times \mathbb{R}P^1$. Then, because of dimensional reason, the geometric finiteness of Π implies that its complexification is also finite. More precisely, fix a $\hat{q} \in \Pi^{-1}(q)$ and work locally in a neighbourhood of \hat{q} . The complexification $(\mathcal{X}_{\mathbb{C}})_{\hat{q}}$ of $\mathcal{X}_{\hat{q}}$ is a complex analytic subset of $(X_{\mathbb{C}} \times \mathbb{CP}^1)_{\hat{q}}$, where by $X_{\mathbb{C}}$ we denote the complexification of X at q . Denote by $\Pi_{\mathbb{C}} : (\mathcal{X}_{\mathbb{C}})_{\hat{q}} \rightarrow (X_{\mathbb{C}})_q$ the projection onto the first factor. Then $\Pi_{\mathbb{C}}^{-1}(q)$ is a complex analytic subset of $(\{q\} \times \mathbb{CP}^1)_{\hat{q}}$ whose real part is reduced to one point and hence is finite itself. Thus $(\Pi_{\mathbb{C}})_{\hat{q}}$ is a finite map. Since a finite blowing-up of a nonsingular complex analytic space is an isomorphism, so is Π . \square

Thus we have shown that σ factors through π as claimed. Then we apply the same procedure to $\hat{\sigma}$. To finish the proof, we note that after a finite number of point blowings-up (more precisely locally finite on S) this process terminates, i.e. the obtained lift is an isomorphism. Indeed, after each blowing-up the number of irreducible components of the exceptional set increases but it cannot be bigger than the number of irreducible (global, analytic) ones of $\sigma^{-1}(p)$. The proof of Theorem 0.2 is complete. \square

3. PROOF OF (1) \Rightarrow (2) OF THEOREM 0.1

Let $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ and $g : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ be real analytic function germs. By Theorem 0.2 there exists a commutative diagram

$$(3.1) \quad \begin{array}{ccccccc} (M, \mu'^{-1}(E)) & \xrightarrow{\mu'} & (\widetilde{\mathbb{R}^2}, E_1) & \xrightarrow{\pi} & (\mathbb{R}^2, 0) & \xrightarrow{f} & \mathbb{R} \\ \Phi \downarrow & & \exists h_1? \downarrow & & h \downarrow & & \parallel \\ (\tilde{M}, \tilde{\mu}'^{-1}(E)) & \xrightarrow{\tilde{\mu}'} & (\widetilde{\mathbb{R}^2}, \tilde{E}_1) & \xrightarrow{\tilde{\pi}} & (\mathbb{R}^2, 0) & \xrightarrow{g} & \mathbb{R} \end{array}$$

where π and $\tilde{\pi}$ is the blowing-up of the origin, μ' and $\tilde{\mu}'$ are compositions of point blowings-up, Φ is an analytic isomorphism and h is a homeomorphism such that $f = g \circ h$.

Proposition 3.1. *There exists a homeomorphism $h_1 : (\widetilde{\mathbb{R}^2}, E_1) \rightarrow (\widetilde{\mathbb{R}^2}, \tilde{E}_1)$, $\tilde{\pi} \circ h_1 = h \circ \pi$, closing the diagram 3.1.*

Proof. (For simplicity we shall drop the germ notation.) By composing $\mu = \pi \circ \mu'$ and $\tilde{\mu} = \tilde{\pi} \circ \tilde{\mu}'$ with additional blowings-up, if necessary, we may assume that they are resolutions of f and g respectively, that is $f \circ \mu$ and $g \circ \tilde{\mu}$ have only normal crossing singularities.

Denote by E_1, \dots, E_k the components of the exceptional divisor of μ , in the order they were created (we shall keep the same notation for the exceptional divisor of a point blowing-up and for its subsequent strict transforms). We show below that among these components, E_1 can be recognised on the dual resolution graph $S(\mu)$ of μ . Hence we conclude that $\Phi(E_1) = \tilde{E}_1$, that is the most difficult step in showing the existence of h_1 .

Lemma 3.2. *Let $m = \text{mult}_0 f$ and suppose that the zero set of the leading homogeneous part f_m of f is not reduced to a point. Then among all divisors of multiplicity m in $\mu^{-1}(0)$, E_1 is completely characterised by the following property:*

- (H) *Either the strict transform of $f^{-1}(0)$ intersects E_1 or there is a connected component of $S(\mu) \setminus E_1$ that does not contain a vertex of multiplicity m .*

(The two conditions of (H) do not exclude themselves and often they both hold for E_1 .)

Proof. We first show that property (H) is hereditary for the divisors of multiplicity m in the following sense. Suppose that $\sigma : N \rightarrow \mathbb{R}^2$ is a real modification and let E be an exceptional divisor of σ . Suppose moreover that $f \circ \sigma$ is a normal crossings in a neighbourhood of E . Let $\sigma' : N' \rightarrow N$ be a blowing-up of $p \in E$. Then we say that (H) is *hereditary* if E satisfies (H) as a divisor of N iff its strict transform satisfies (H) as a divisor of N' . In our case one may check the heredity easily by inspection, since either $\text{mult}_p f \circ \sigma = m$ or E intersects at p another divisor or the strict tranform of $f^{-1}(0)$ and then $\text{mult}_p f \circ \sigma > m$.

In general, let E be a component of the exceptional divisor of a real modification $\sigma : N \rightarrow \mathbb{R}^2$. We call a point $p \in E$ *simple* (with respect to f) if $\text{mult}_p f \circ \sigma = \text{mult}_E(f)$. A divisor $E \neq E_1$ of multiplicity m can be created only by blowing-up a simple point on another divisor of multiplicity m . Therefore for such a divisor $f \circ \sigma$ is always normal crossings in a neighbourhood of E and does not satisfy (H) at the moment it is created, and hence, by heredity, never.

By assumption on f_m , the divisor E_1 , when created by the first blowing-up π , contains some points of multiplicity higher than m . If its strict transform in M does not intersect the strict transform of $f^{-1}(0)$, it means that all the points of multiplicity higher than m on $\pi^{-1}(0)$ have been blown-up. Each of such blowing-up produces a divisor of multiplicity higher than m and a connected component of $S(\mu) \setminus E_1$ that does not contain a divisor of multiplicity m . Blowing-up points on the divisors on this component cannot produce new divisors of multiplicity m . This ends the proof of lemma. \square

Suppose that $f_m^{-1}(0) = \{0\}$. Then the blowing-up of the origin $\pi : \widetilde{\mathbb{R}^2} \rightarrow \mathbb{R}^2$ resolves f . The modification μ is the composition of π and finitely many point blowings-up. To each component E of $\mu^{-1}(0)$ we associate the following number

$$(3.2) \quad \varepsilon_\mu(E) = \sum_{E' \in \mathcal{E}_\mu} \frac{m(E')}{m(E)} (E', E)_\mu,$$

where by \mathcal{E}_μ we denote the set of all exceptional divisors of μ and $(E, E')_\mu$ is defined by the following rule

$$(3.3) \quad (E, E')_\mu = \begin{cases} 1 & \text{if } E \neq E' \text{ and } E \cap E' \neq \emptyset \\ 0 & \text{if } E \cap E' = \emptyset \\ p_\mu(E) & \text{if } E = E', \end{cases}$$

where the parity $p_\mu(E)$ of E equals 0 if E admits an orientable neighbourhood in M , otherwise $p_\mu(E) = 1$. That is $(E, E)_\mu \pmod{2}$ equals the intersection number of E and E' . We can detect E_1 among all the other components thanks to the following lemma.

Lemma 3.3. *Under the above assumptions $\varepsilon_\mu(E) \in \mathbb{Z}$ and*

$$\varepsilon_\mu(E_1) \equiv 1 \pmod{2}, \quad \varepsilon_\mu(E) \equiv 0 \pmod{2} \text{ for } E \neq E_1.$$

Proof. We check how $\varepsilon_\mu(E)$ changes under blowings-up. Let $\sigma : N \rightarrow M$ be a blowing-up of $p \in E$ and $\mu' = \mu \circ \sigma$. If p is a simple point of E then a new divisor of multiplicity $m(E)$ is produced and the parity of E changes. Both these events affect the sum in (3.2) by adding ± 1 . Similarly, if $p \in E \cap E'$ then a new divisor E'' of multiplicity $m(E) + m(E')$ is created. Then $(E, E'')_{\mu'} = 1$ and $(E, E')_{\mu'} = 0$. Moreover, the parity of E is reversed. Therefore

$$\frac{m(E'')}{m(E)}(E, E'')_{\mu'} + \frac{m(E)}{m(E)}(E, E)_{\mu'} \equiv \frac{m(E')}{m(E)}(E, E')_{\mu} + \frac{m(E)}{m(E)}(E, E)_{\mu} \pmod{2}$$

that shows that $\varepsilon_\mu(E) \pmod{2}$ does not depend on μ . Then we compute $\varepsilon_\mu(E) \pmod{2}$ at the moment E is created. If $E = E_1$ we take $\mu = \pi$ and get

$$\varepsilon_\pi(E_1) = 1,$$

as claimed. If $E \neq E_1$ we denote by $\sigma : N \rightarrow \mathbb{R}^2$ the real modification that has E as the last created divisor. Then E is the exceptional divisor of the blowing-up of either a simple point on a divisor E_i , or of the intersection point of two distinct divisors E_i, E_j . In the former case

$$\varepsilon_\sigma(E) = \frac{m(E_i)}{m(E)}(E_i, E)_\sigma + \frac{m(E)}{m(E)}(E, E)_\sigma = 1 + 1 \equiv 0 \pmod{2}.$$

In the latter case

$$\begin{aligned} \varepsilon_\sigma(E) &= \frac{m(E_i)}{m(E)}(E_i, E)_\sigma + \frac{m(E_j)}{m(E)}(E_j, E)_\sigma + \frac{m(E)}{m(E)}(E, E)_\sigma \\ &= \frac{m(E_i)}{m(E)} + \frac{m(E_j)}{m(E)} + 1 \equiv 0 \pmod{2}. \end{aligned}$$

This ends the proof of lemma. □

We continue the proof of Proposition 3.1. If $f_m^{-1}(0) = \{0\}$, then E_1 does not satisfy the property (H) of Lemma 3.2. Therefore there is no exceptional divisor of μ of multiplicity m that satisfies this property. Consequently the same is true for \tilde{E}_1 and $\tilde{\mu}$. Therefore $g_m^{-1}(0) = \{0\}$ and we may conclude by Lemma 3.3 that $\Phi(E_1) = \tilde{E}_1$.

Similarly if E_1 satisfies the property (H) then so does \tilde{E}_1 and $\Phi(E_1) = \tilde{E}_1$ by Lemma 3.2.

Denote by $F \subset \pi^{-1}(0)$, resp. $\tilde{F} \subset \tilde{\pi}^{-1}(0)$, the image of the other exceptional divisors of μ , resp. $\tilde{\mu}$. Thus F and \tilde{F} are finite. For each $p \in F$, $(\mu')^{-1}(p)$ is the union of all divisors in a connected component of $S(\mu) \setminus E_1$. Therefore Φ sends $(\mu')^{-1}(p)$ onto $(\tilde{\mu}')^{-1}(q)$ for a unique $q \in \tilde{F}$. Moreover Φ induces a homeomorphism of $E_1 \setminus F$ onto $\tilde{E}_1 \setminus \tilde{F}$. Since both μ' and $\tilde{\mu}'$ are proper this is sufficient to conclude that Φ induces a homeomorphism $h_1 : (\mathbb{R}^2, E_1) \rightarrow (\mathbb{R}^2, \tilde{E}_1)$, as claimed. This ends the proof of Proposition 3.1. □

Lemma 3.4. *If $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ is normal crossing and f and $g : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ are blow-analytically equivalent then g is also normal crossing (with the same exponents).*

Proof. We shall only consider the case $f = x^a y^b$, $a > 0, b > 0$, the case when $f = x^a$ is easier. Then the zero set of g contains two real analytic curves, one with generic multiplicity a and the other one with generic multiplicity b . Therefore

$$g = (g_1)^a (g_2)^b g_3.$$

Since the multiplicity is a blow-analytic invariant, $\text{mult}_0 g = a + b$. Consequently g_3 is a unit and g_1, g_2 are regular germs.

The zero set $f^{-1}(0)$ of f divides a neighbourhood of 0 in \mathbb{R}^2 into 4 sectors and each of them contains a half branch of real analytic curve $\gamma(t) : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ on which $f \circ \gamma(t)$ has order $a + b$. The invariance of the refined Fukui invariant, cf. subsection 1.2, shows that the same is true for the four sectors of the complement of $g^{-1}(0)$. This implies that the zero sets $g_1^{-1}(0)$ and $g_2^{-1}(0)$ are transverse and ends the proof of lemma. \square

Suppose that the blow-analytic equivalence of f and g be given (0.1). Performing additional point blowings-up, if necessary, we may assume that both $f \circ \mu$ and $g \circ \tilde{\mu}$ are normal crossings. By Theorem 0.2, μ is the composition of a sequence of points blowings-up

$$(3.4) \quad M = M_k \xrightarrow{b_k} M_{k-1} \xrightarrow{b_{k-1}} \dots \xrightarrow{b_2} M_1 = \widetilde{\mathbb{R}^2} \xrightarrow{b_1=\pi} \mathbb{R}^2$$

The order of choice of centres of the blowings-up is not unique, if we have to blow-up two different points on $\widetilde{\mathbb{R}^2}$ we may do it in any order. Suppose that we first blow-up only singular points (in the sense of non-normal crossing) of the total transforms of f . The composition of these, say $s(f)$, blowings-up is the minimal resolution of f . Then, using repeatedly Proposition 3.1 we obtain

$$(3.5) \quad \begin{array}{ccccccc} M & \xrightarrow{b_k} & M_{k-1} & \xrightarrow{b_{k-1}} & \dots & \xrightarrow{b_2} & M_1 \xrightarrow{b_1} \mathbb{R}^2 \xrightarrow{f} \mathbb{R} \\ \Phi \downarrow & & \downarrow h_{k-1} & & & \downarrow h_1 & \downarrow h & \parallel \\ \tilde{M} & \xrightarrow{\tilde{b}_k} & \tilde{M}_{k-1} & \xrightarrow{\tilde{b}_{k-1}} & \dots & \xrightarrow{\tilde{b}_2} & \tilde{M}_1 \xrightarrow{\tilde{b}_1} \mathbb{R}^2 \xrightarrow{g} \mathbb{R} \end{array}$$

By Lemma 3.4, whenever $p \in M_i$ is singular (in the sense of non-normal crossing) for the total transform of f then $h_i(p) \in \tilde{M}_i$ is also singular for the total transform of g , and vice versa. Consequently, the composition of the first $s(f)$ blowings-up of the lower row of (3.5) is the minimal resolution of g .

This ends the proof of (1) \Rightarrow (2) of Theorem 0.1.

4. REAL ANALYTIC DEMI-BRANCHES

By a (*parametrised*) *real analytic arc* at $0 \in \mathbb{R}^2$ we mean an analytic non-constant germ $\gamma(t) : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$. We consider two such arcs $\gamma, \tilde{\gamma}$ *equivalent* if there is an analytic orientation preserving isomorphism germ $\sigma : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ such that $\tilde{\gamma} = \gamma \circ \sigma$. The equivalence classes will be called *real analytic arcs* at $0 \in \mathbb{R}^2$.

A real analytic arc γ will be called *reduced* if it cannot be written down as $\gamma(t) = \eta(t^k)$, where $k > 1$ and η is a real analytic arc. If γ is reduced then it is injective as a map and its image is an irreducible analytic germ of dimension 1.

Remark 4.1. If $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ is a blow-analytic homeomorphism and $\gamma(t) : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ is reduced then so is $h \circ \gamma$.

By a *real analytic demi-branch at $0 \in \mathbb{R}^2$* we mean a parametrised real analytic arc $\gamma(t) : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ restricted to $t \geq 0$. We again identify γ and $\tilde{\gamma}$ if $\tilde{\gamma} = \gamma \circ \sigma$ for an orientation preserving analytic isomorphism $\sigma : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$. By Puiseux Theorem each reduced real analytic demi-branch can be expressed, after a coordinate change in $(\mathbb{R}^2, 0)$, as $\gamma(t) = (\lambda(t^m), t^m)$, where λ is a fractional power series such that $\text{ord}_0 \lambda(y) \geq 1$. If this is the case then we say, for short, that the demi-branch γ is given by

$$(4.1) \quad x = \lambda(y) = a_{m_1/m} y^{m_1/m} + a_{m_2/m} y^{m_2/m} + \cdots, \quad y \geq 0$$

where $m \leq m_1 < m_2 < \cdots$ are positive integers, having no common divisor. Sometimes we also assume that $m_1 > m$ and that m_1 does not divide m . For a single arc this is always possible after another analytic coordinate change.

By the *tangent direction at 0 of a demi-branch $\gamma(t)$* we mean $\lim_{t \rightarrow 0} \frac{\gamma(t)}{\|\gamma(t)\|} \in S^1$. Let γ_1, γ_2 be reduced real analytic demi-branches tangent at 0. Suppose that γ_1, γ_2 are given, in the same system of coordinates by Puiseux series λ_1 , resp. λ_2 . We define the *contact order of γ_1 and γ_2* as

$$O(\gamma_1, \gamma_2) := \text{ord}_0 (\lambda_1 - \lambda_2)(y).$$

If γ_1, γ_2 have distinct tangent directions at 0 then we define $O(\gamma_1, \gamma_2) = 1$. The order contact between two real analytic demi-branches is well-defined, that is it is independent of their Puiseux presentations of the form (4.1).

The Puiseux pairs of a reduced real analytic demi-branch γ are pairs of relatively prime positive integers $(n_1, d_1), \dots, (n_q, d_q)$, $d_i > 1$ for $i = 1, \dots, q$, $\frac{n_1}{d_1} < \frac{n_2}{d_1 d_2} < \cdots < \frac{n_q}{d_1 \dots d_q}$, such that

$$(4.2) \quad \begin{aligned} \lambda(y) &= \sum_{\alpha} a_{\alpha} y^{\alpha} = \\ &= \sum_{j=1}^{[n_1/d_1]} a_j y^j + \sum_{j=n_1}^{[n_2/d_2]} a_{j/d_1} y^{j/d_1} + \sum_{j=n_2}^{[n_3/d_3]} a_{j/d_1 d_2} y^{\frac{j}{d_1 d_2}} + \cdots + \sum_{j=n_q}^{\infty} a_{j/d_1 d_2 \dots d_q} y^{\frac{j}{d_1 d_2 \dots d_q}} \end{aligned}$$

and $a_{n_i/d_1 \dots d_i} \neq 0$ for $i = 1, \dots, q$. Following [33] chapter 3 we call the integers $m; \beta_1 = n_1 d_2 \dots d_q, \beta_2 = n_2 d_3 \dots d_q, \dots, \beta_q = n_q$ the *Puiseux characteristic sequence of γ* . We will call $m = d_1 d_2 \dots d_q$ the *multiplicity of γ* . The coefficients $A_i(\gamma) := a_{n_i/d_1 \dots d_i} = a_{\beta_i/m}$ for $i = 1, \dots, q$ will be called the *characteristic coefficients of γ* .

Proposition 4.2. *Let γ be a reduced real analytic demi-branch of multiplicity m . Then the signs of characteristic coefficients are well-defined, they are independent of the Puiseux presentation $\gamma(t) = (\lambda(t^m), t^m)$ in an oriented system of coordinates at $(\mathbb{R}^2, 0)$.*

Proof. Let $\gamma(t) = (\lambda(t^m), t^m)$. Fix $i = 1, \dots, q$ and consider a new demi-branch γ_i defined in the same system of coordinates by

$$\gamma_i(t) = (\lambda_i(t^m), t^m), \quad \lambda_i(y) = \sum_{\alpha < \beta_i/m} a_{\alpha} y^{\alpha}.$$

The order of contact between γ and γ_i equals exactly β_i/m and (n_i, d_i) is not the i -th Puiseux pair of γ_i . These two properties characterise γ_i "up to terms of order higher than β_i/m ". Since $\beta_i/m > 1$ both demi-branches are tangent. The i -th characteristic coefficients of γ is positive if and only if γ follows γ_i in the clock-wise direction in the orientation of $(\mathbb{R}^2, 0)$ induced by the coordinate system x, y . \square

4.1. Effect of a blowing-up. Let $\gamma(t)$ be a reduced real analytic demi-branch. Denote by $\tilde{\gamma}$ the strict transform of γ by the blowing-up of the origin $\pi : \widetilde{\mathbb{R}^2} \rightarrow \mathbb{R}^2$. Then $\tilde{\gamma}$ is a reduced real analytic demi-branch based at a point of $\widetilde{\mathbb{R}^2}$ that we denote by $\tilde{0}$. The Puiseux characteristic sequence of $\tilde{\gamma}$ is the following, cf. [33] Theorem 3.5.5,

$$(4.3) \quad \begin{array}{ll} (m; \beta_1 - m, \beta_2 - m, \dots, \beta_q - m) & \text{if } \beta_1 > 2m \\ (\beta_1 - m; m, \beta_2 - \beta_1 + m, \dots, \beta_q - \beta_1 + m) & \text{if } \beta_1 < 2m, (\beta_1 - m) \nmid m \\ (\beta_1 - m; \beta_2 - \beta_1 + m, \dots, \beta_q - \beta_1 + m) & \text{if } \beta_1 < 2m, (\beta_1 - m) \mid m \end{array}$$

Consider $(\mathbb{R}^2, 0)$ oriented. Put an orientation on $(\widetilde{\mathbb{R}^2}, \tilde{0})$ so that π preserves orientation at the points of $\tilde{\gamma}$. (This orientation depends on the tangent direction of γ at 0, if we take consider the demi-branches with the opposite tangent directions we get the opposite orientations on $(\widetilde{\mathbb{R}^2}, \tilde{0})$).

Proposition 4.3. *The signs of the characteristic coefficients \tilde{A}_i of $\tilde{\gamma}$ are :*

- (i) Case 1 of (4.3): $\text{sign } \tilde{A}_i = \text{sign } A_i, \quad i = 1, \dots, q.$
- (ii) Case 2 of (4.3): $\text{sign } \tilde{A}_1 = -\text{sign } A_1$ and $\text{sign } \tilde{A}_i = \text{sign } A_i, \quad i = 2, \dots, q.$
- (iii) Case 3 of (4.3): $\text{sign } \tilde{A}_i = \text{sign } A_{i+1}, \quad i = 1, \dots, q-1.$

Proof. We may choose a system of coordinates x, y at $0 \in \mathbb{R}^2$ so that $\gamma(t) = (\lambda(t^m), t^m)$ and λ is given by (4.1) with $m_1 = \beta_1$. Then $\tilde{x} := x/y, \tilde{y} := y$, is a system of coordinates at $\tilde{0}$ on $\widetilde{\mathbb{R}^2}$ and $\tilde{\gamma}$ is given by $\tilde{\gamma}(t) = (\tilde{\lambda}(t^m), t^m)$, where

$$\tilde{\lambda}(\tilde{y}) = a_{m_1/m} \tilde{y}^{(\beta_1-m)/m} + a_{m_2/m} \tilde{y}^{(m_2-m)/m} + \dots, \quad \tilde{y} \geq 0$$

In Case 1 of (4.3), $(\beta_1 - m)/m > 1$ and the characteristic coefficients satisfy $\tilde{A}_i = A_i$.

In Case 2 of (4.3) we reparametrise $\tilde{\gamma}$ as follows. Let

$$\tilde{\lambda}(t^m) = A_1 t^{\beta_1-m} + \dots = A_1 (\tilde{t}(t))^{\beta_1-m}, \quad \tilde{t}(t) = t + \dots.$$

In the coordinates $\tilde{y}, \tilde{x}/A_1$

$$\tilde{\gamma}(\tilde{t}) = (\delta(\tilde{t}^{\beta_1-m}), \tilde{t}^{\beta_1-m}),$$

where in $\delta(\tilde{t}^{\beta_1-m}) = (t(\tilde{t}))^m = \tilde{t}^m + \dots$.

The following lemma can be obtained by direct computation.

Lemma 4.4. *Let the series $t(\tilde{t}) = \tilde{t} + \sum_{k=2}^{\infty} b_k \tilde{t}^k$ be defined by*

$$(4.4) \quad t^{\beta_1-m} + \sum_{j=\beta_1+1}^{\infty} a_j t^{j-m} = (\tilde{t}(t))^{\beta_1-m}.$$

Then for each $k \geq 2$

$$b_k = \frac{-a_{k+\beta_1-1}}{(\beta_1 - m)} + P_k(a_{\beta_1+1}, \dots, a_{k+\beta_1-2}),$$

where P_k is a polynomial. Moreover, if the coefficient at $(a_{\beta_1+j_1})^{\alpha_1} \cdots (a_{\beta_1+j_r})^{\alpha_r}$ in P_k is non-zero then $k-1 = \alpha_1 j_1 + \cdots + \alpha_r j_r$. \square

By lemma the coefficient at $\tilde{t}^{\beta_i-\beta_1+1}$, $i > 1$, in $t(\tilde{t})$ equals $\frac{-A_i}{(\beta_1-m)A_1}$. Consequently the coefficient at $\tilde{t}^{\beta_i-\beta_1+m}$, $i > 1$, in $\delta(\tilde{t}^{\beta_1-m})$ equals

$$\tilde{A}_i = \frac{-mA_i}{(\beta_1-m)A_1}.$$

The coordinates $\tilde{y}, \tilde{x}/A_1$ give the chosen orientation on $(\widetilde{\mathbb{R}^2}, \tilde{0})$ iff $A_1 < 0$. This shows Proposition 4.3 Case 2.

The proof of Case 3 of (4.3) is similar. \square

Proposition 4.5. *Let γ_1, γ_2 be reduced real analytic demi-branches tangent at 0 and denote by $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ their strict transforms by the blowing-up of the origin.*

Denote by m the multiplicity of γ_1 , and if $m > 1$, i.e. if γ_1 is not smooth, by β_1 the second exponent of the characteristic sequence of γ_1 . The corresponding exponents of γ_2 will be denoted by n and α_1 . Then

- (1) *If $\beta_1/m > 2$ or $m = 1$, and $\alpha_1/n > 2$ or $n = 1$, then*

$$O(\tilde{\gamma}_1, \tilde{\gamma}_2) = O(\gamma_1, \gamma_2) - 1.$$

- (2) *If $\beta_1/m < 2$, and $\alpha_1/n > 2$ or $n = 1$, then*

$$O(\tilde{\gamma}_1, \tilde{\gamma}_2) = 1.$$

- (3) *If $\beta_1/m < 2$, $\alpha_1/n < 2$ and the first characteristic coefficients of γ_1 and γ_2 are of opposite sign then*

$$O(\tilde{\gamma}_1, \tilde{\gamma}_2) = 1.$$

- (4) *If $\beta_1/m < \alpha_1/n < 2$ and the first characteristic coefficients of γ_1 and γ_2 are of same sign then*

$$O(\tilde{\gamma}_1, \tilde{\gamma}_2) = \frac{n}{\alpha_1 - n}$$

- (5) *If $\beta_1/m = \alpha_1/n < 2$ and the first characteristic coefficients of γ_1 and γ_2 are of same sign then*

$$O(\tilde{\gamma}_1, \tilde{\gamma}_2) = \frac{m}{\beta_1 - m} O(\gamma_1, \gamma_2) - 1.$$

Proof of Proposition 4.5. Choose a system of coordinates so that $\gamma_1(t) = (\lambda_1(t^m), t^m)$, $\gamma_2(t) = (\lambda_2(t^n), t^n)$, where λ_1 and λ_2 are as in (4.1). Then Cases (1) and (2) are easy. In Case (1) all exponents in λ_1 and λ_2 are at least 2 and the claim easily follows. In Case (2) $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are not tangent.

Suppose that $\beta_1/m < 2$ and $\alpha_1/n < 2$ and write

$$\begin{aligned} \lambda_1(y) &= A_1 y^{\beta_1/m} + \cdots, \\ \lambda_2(y) &= B_1 y^{\alpha_1/n} + \cdots. \end{aligned}$$

Then both $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are tangent to the exceptional divisor. If the signs of A_1 and B_1 are opposite then $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ have opposite tangent directions. This shows (3).

Suppose now that $A_1 > 0$ and $B_1 > 0$. In order to compare $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, as in the proof of Proposition 4.3, we have to pass to the coordinates $\tilde{y} := y, \tilde{x} := x/y$, and reparametrise the demi-branches so we express $\tilde{\gamma}_i$ in terms of $\delta_i(\tilde{x})$, $i = 1, 2$. The leading exponent of $\delta_1(\tilde{x})$, resp. $\delta_2(\tilde{x})$, is $\frac{m}{\beta_1 - m}$, resp. $\frac{n}{\alpha_1 - n}$. This shows (4) if $\beta_1/m < \alpha_1/n$.

It remains to consider the case $\beta_1/m = \alpha_1/n$. If $A_1 \neq B_1$, i.e. $\beta_1/m = \alpha_1/n = O(\gamma_1, \gamma_2)$, then the leading coefficients of $\delta_1(\tilde{x})$ and $\delta_2(\tilde{x})$ are different. This completes the remaining case of (4). If $A_1 = B_1$ then, as in the proof of Proposition 4.3, we use the coordinates $\tilde{y}, \tilde{x}/A_1$. Let $\xi = O(\gamma_1, \gamma_2)$. Then the computation of the proof of Proposition 4.3 for ξ in place of β_i/m and shows that the first different coefficients are the ones at $\tilde{x}^{(\beta_i - \beta_1 + m)/(\beta_1 - m)}$ as claimed in (5). \square

4.2. The symmetric demi-branch. Let $\gamma(t) = (\lambda(t^m), t^m)$, $t \geq 0$, be a reduced real analytic demi-branch, where $\lambda(y)$ is given by (4.2). The symmetric demi-branch γ_- of γ is obtained by replacing t by $-t$. If m is odd it corresponds to replacing y by $-y$ in (4.1). Both demi-branches have the same Puiseux characteristic sequences and if m is odd the signs of characteristic coefficients are also the same.

Suppose now that m is even. Then γ_- and γ are tangent and we may compare their forms (4.1) in the same system of coordinates. Then

$$(4.5) \quad \lambda_-(y) = \sum_j a_{j/m} (-1)^j y^{j/m}.$$

Let $m = 2^s(2l + 1)$, $s > 0$, and let d_p be the last even number in the sequence d_1, \dots, d_q (thus β_p, \dots, β_q are odd). Then the characteristic coefficients of γ_- are $A_i(\gamma)$ if $i < p$ and $-A_i(\gamma)$ if $i \geq p$. The order contact between both demi-branches is $O(\gamma, \gamma_-) = \beta_p/m$.

5. CASCADE BLOW-ANALYTIC HOMEOMORPHISMS

Recall that a blow-analytic homeomorphism $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ is called *cascade* if there exists a commutative diagram

$$(5.1) \quad \begin{array}{ccccccc} M & \xrightarrow{b_k} & M_{k-1} & \xrightarrow{b_{k-1}} & \dots & \xrightarrow{b_2} & M_1 & \xrightarrow{b_1} & \mathbb{R}^2 \\ \Phi \downarrow & & \downarrow h_{k-1} & & & & \downarrow h_1 & & \downarrow h \\ M' & \xrightarrow{b'_k} & M'_{k-1} & \xrightarrow{b'_{k-1}} & \dots & \xrightarrow{b'_2} & M'_1 & \xrightarrow{b'_1} & \mathbb{R}^2, \end{array}$$

where b_i and b'_i are point blowings-up, h_i are homeomorphisms induced by an analytic isomorphism Φ .

If a real analytic arc $\gamma : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ is injective then its image is an irreducible real analytic curve germ $(X, 0) \subset (\mathbb{R}^2, 0)$ of dimension 1. Let $(X_1, 0), (X_2, 0)$ be two such curve germs. If $(X_1, 0), (X_2, 0)$ are distinct then we define the *intersection number* $(X_1, X_2)_0 \in \mathbb{N}$ by taking the intersection number of the complexifications. By a *real analytic 1-cycle at $0 \in \mathbb{R}^2$* we mean a formal sum with integer coefficients of such curve germs. Thus, by additivity we may consider the intersection number at 0 between two real analytic 1-cycles whose supports do intersect only at the origin.

Theorem 5.1. *Let $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be a cascade blow-analytic homeomorphism. Then h preserves:*

- (a) *The tangency of real analytic arcs and real analytic demi-branches.*
- (b) *The Puiseux characteristics sequence and hence the multiplicity of a reduced real analytic demi-branch.*
- (c) *The order of contact between two reduced real analytic demi-branches.*
- (d) *The intersection number between two real analytic 1-cycles at 0 (of supports intersecting only at the origin).*
- (e) *If, moreover, h is orientation preserving, then it preserves the signs of characteristic coefficients of a reduced real analytic demi-branch.*

Proof. The proof is by induction on the number k of point blowings-up in (5.1). We assume that h preserves the orientation. (a) is obvious by the existence of h_1 .

Let γ be a reduced real analytic arc and let $(X_\gamma, 0) \subset (\mathbb{R}^2, 0)$ by the image of γ . Then the multiplicity of γ equals the multiplicity of X_γ at 0 and this equals the intersection number of the strict transform of $(X_\gamma, 0)$ by b_1 with the exceptional divisor. Thus the invariance of the multiplicity follows from the inductive assumption.

Now we show (d). Let X_1, X_2 be two distinct irreducible curves, images of γ_1, γ_2 . We suppose that X_1, X_2 are tangent at the origin and denote by \tilde{X}_1, \tilde{X}_2 their respective strict transforms intersecting at $\tilde{0}$. Then $(X_1, X_2)_0 = m_0(X_1)m_0(X_2) + (\tilde{X}_1, \tilde{X}_2)_{\tilde{0}}$ and (d) follows by induction.

Let γ be a reduced real analytic demi-branch and denote by $\tilde{\gamma}$ its strict transform by π_1 . Then, by (4.3), the Puiseux characteristics of γ can be expressed in terms of the multiplicity of γ and the Puiseux characteristics of $\tilde{\gamma}$. Thus (b) follows by the inductive assumption and the invariance of the multiplicity.

Let γ is a reduced real analytic demi-branch of multiplicity $m > 1$ such that the second exponent β_1 of the characteristic sequence of γ satisfies $\beta_1 < 2m$. Let γ_1 be any smooth real analytic demi-branch tangent to γ . Then the first characteristic coefficients of γ is positive if and only if γ follows γ_1 in the clock-wise direction and therefore its sign is preserved by h (thus we have shown a special case of (e)).

Consequently (c) follows from Proposition 4.5, (a), (b), the just proven special case of (e), and the inductive assumption.

The general case of (e) now follows from the same argument that we used to show Proposition 4.2. \square

Remark 5.2. Let $f_t : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}, 0)$, $t \in \mathbb{R}$, be the Briançon-Speder family defined by $f_t(x, y, z) = z^5 + tzy^6 + y^7x + x^{15}$. Although f_0 and f_{-1} are blow-analytically equivalent, any blow-analytic homeomorphism that gives the blow-analytic equivalence between them does not preserve the order of contact between some analytic arcs contained in $f_0^{-1}(0)$, see [17].

Corollary 5.3. *Let $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be a cascade blow-analytic homeomorphism. Then there exists a constant $C > 0$ such that for all (x, y) close to the origin*

$$C^{-1}\|(x, y)\| \leq \|h(x, y)\| \leq C\|(x, y)\|.$$

Proof. This follows from the invariance of multiplicity. Indeed, by the curve selection lemma, it suffices to check it on a real analytic demi-branch $\gamma(t)$ and we may assume that γ is reduced. Then $\|\gamma(t)\|$ is of size t^m , where m denotes the multiplicity of γ . \square

We also note the following obvious property of cascade blow-analytic homeomorphisms.

Proposition 5.4. *Let $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be a cascade blow-analytic homeomorphism given by (5.1). Denote $\mu = b_1 \circ b_2 \circ \dots \circ b_k$ and $\mu' = b'_1 \circ b'_2 \circ \dots \circ b'_k$. Then the Jacobian determinants of $\mu' \circ \Phi$ and μ , defined in any local system of coordinates on M , are equal up to a multiplication by a unit. As a consequence there exists a constant $C > 0$ such that for $(x, y) \neq (0, 0)$ and close to the origin*

$$C^{-1} \leq \text{Jac}(h)(x, y) \leq C.$$

\square

6. REAL TREE MODEL

6.1. Tree model of a two variable complex analytic function germ. Let $f(x, y)$ be a complex analytic function germ of multiplicity m and *mini-regular* in x , that is

$$f(x, y) = u(x, y)(x^m + \sum_{i=1}^m a_i(y)x^{m-i}),$$

where $m = \text{mult}_0 f$, u, a_i are analytic and $u(0, 0) \neq 0$. Let $x = \lambda_i(y)$, $i = 1, \dots, m$, be the complex Newton-Puiseux roots of f . Define the *contact order* of λ_i and λ_j as

$$O(\lambda_i, \lambda_j) := \text{ord}_0(\lambda_i - \lambda_j)(y).$$

Let $h \in \mathbb{Q}$. We say that λ_i, λ_j are *congruent modulo h^+* if $O(\lambda_i, \lambda_j) > h$.

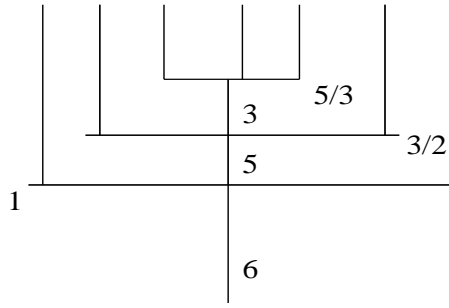
The tree model $T(f)$ of f is defined as follows, see [23] for details. First, draw a vertical line segment as the *main trunk* of the tree. Mark $m = \text{mult}_0 f(x, y)$ alongside the trunk to indicate that m roots are bundled together.

Let $h_0 := \min\{O(\lambda_i, \lambda_j) | 1 \leq i, j \leq m\}$. Then draw a bar, B_0 , on top of the main trunk. Call $h(B_0) := h_0$ the *height* of B_0 and mark it on the tree.

The roots are divided into equivalence classes modulo h_0^+ . We then represent each equivalence class by a vertical line segment drawn on top of B_0 . Each is called a *trunk*. If a trunk consists of s roots we say it has *multiplicity* s , and mark s alongside (usually if $s = 1$ we do not mark it).

Now, the same construction is repeated recursively on each trunk, getting more bars, then more trunks, etc.. The height of each bar and the multiplicity trunk, are defined likewise. Each trunk has a unique bar on top of it. The construction terminates at the stage where the bar has infinite height, that is on top of a trunk that contains a single, maybe multiple, root of f .

Example 6.1. The tree model of $f(x, y) = (x + y)(x^2 + y^3)(x^3 - y^5)$.



The sets of roots corresponding to trunks are called in [29] *bunches*. Thus each bunch A is the set of roots going through a unique bar $B(A)$, one may say in this case that A is the bunch bounded by $B(A)$. In this way we establish a one-to-one correspondence between trunks, bars, and bunches of roots.

Fix a bunch A , $B = B(A)$, with finite height denoted $h(A)$ or $h(B)$. Take a root $\lambda_i(y)$ of A . Let $\lambda_A(y)$ denote $\lambda_i(y)$ with all terms y^e , $e \geq h$, omitted. Clearly, λ_A depends only on A , not on the choice of $\lambda_i \in A$. We can then write for each $\lambda_i(y) \in A$

$$(6.1) \quad \lambda_i(y) = \lambda_A(y) + c_i y^{h(A)} + \cdots, \quad c_i \in \mathbb{C}.$$

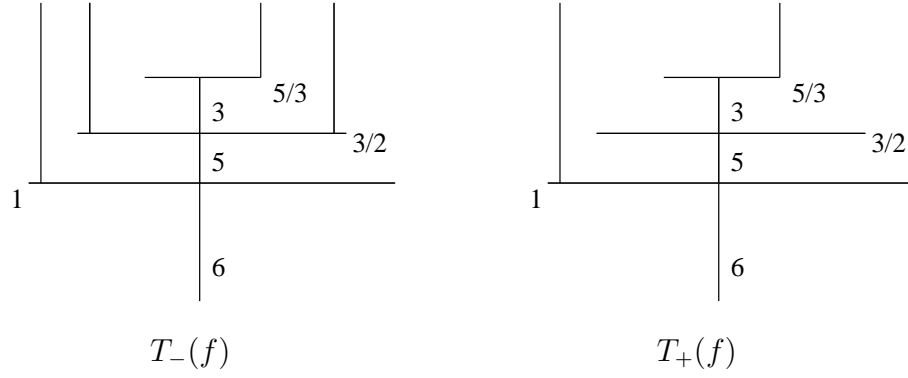
Remark 6.2. The fractional power series $\lambda_i(y)$ are well-defined only up to the action of a group of roots of unity. One may make them well-defined by fixing the argument of $y \in \mathbb{C}$, for instance $y \in \mathbb{R}, y > 0$, cf. [33] p. 98.

Fix a bunch A . Suppose that the denominator of $h(A)$ does not divide the common denominator of exponents of λ_A . It means that these roots of A for which $c_i \neq 0$ get a new Puiseux pair at the bar $B(A)$ and those with $c_i = 0$ do not. Formally, for a particular root, the tree does not contain the information what is the coefficient c_i at $y^{h(A)}$ or even whether $c_i \neq 0$. Nevertheless, by counting the number of subbunches of A , one can read from the tree whether there exists a sub-bunch that does not take a new Puiseux pair at $B(A)$. If it exists it must be unique.

6.2. Real part of tree model. Suppose that $f(x, y)$ is real analytic. Consider the Newton-Puiseux roots as arcs $x = \lambda_i(y)$ defined for $y \in \mathbb{R}, y \geq 0$. The complex conjugation acts on the roots, and hence on the tree model $T(f)$. A bunch A of $T(f)$ is called *real* if it is stable by complex conjugation, or equivalently if λ_A is real. A bar or a trunk is *real* if and only if so is the corresponding bunch. After [29] the conjugation invariant part of $T(f)$, that we denote by $T_+(f)$, is called *the real part of $T(f)$* .

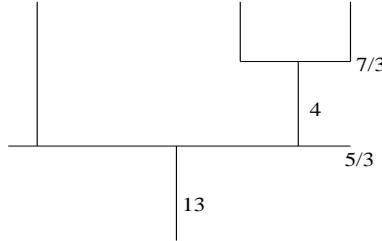
Similarly, by fixing $y \leq 0$, we may define $T_-(f)$. We may identify $T_+(f)$ and $T_-(f)$ if all denominators of the exponents of (6.1) are odd. But in general $T_+(f)$ and $T_-(f)$ are different.

Example 6.3. We draw $T_+(f)$ and $T_-(f)$ for $f(x, y) = (x + y)(x^2 + y^3)(x^3 - y^5)$ of Example 6.1



Remark 6.4. In [29] the authors define also the almost real bunches. For f real analytic the notions of a real bunch and of an almost real bunch coincide.

Remark 6.5. The real part of tree model, even if all denominators in λ_i are odd, does not determine the Newton-Puiseux pairs of the real roots. Indeed, Let $f(x, y) = x(x^3 - y^5)((x^3 - y^5)^3 - y^{17})$, $g(x, y) = x(x^3 + y^5)(x^3 - y^7)(x^6 + x^3y^5 + y^{10})$. All singular roots of f have a single Puiseux pair $(5, 3)$ and two of such roots have contact order $7/3$. The singular roots of g have also a single Puiseux pair, either $(5, 3)$ or $(7, 3)$. Then the real parts $T_+(f) = T_+(g)$ of the tree models coincide, but the real tree models of f and g , see next subsection, are different. As a consequence of Theorem 0.1 we see that f and g are not blow-analytically equivalent. See also example 6.7



6.3. Real tree model of f . Let $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ be a real analytic function germ. Fix v a unit vector of \mathbb{R}^2 . Fix any local system of coordinates x, y such that

- $f(x, y)$ is mini-regular in x
- v is of the form (v_1, v_2) with $v_2 > 0$.

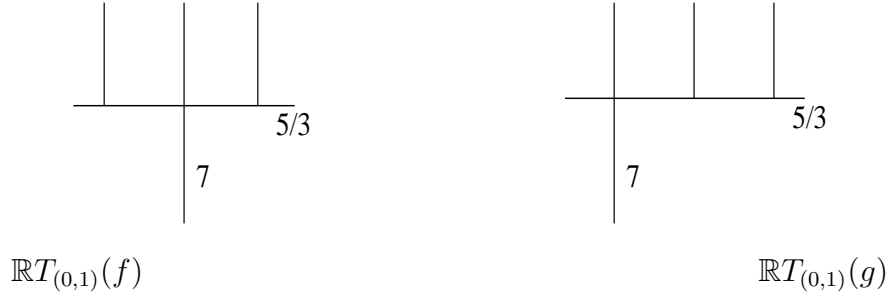
Consider the Newton-Puiseux roots as arcs $x = \lambda_i(y)$ defined for $y \in \mathbb{R}, y \geq 0$.

We define *the real tree model of f relative to v* , and denote it by $\mathbb{R}T_v(f)$, as the part of $T_+(f)$ consisting only of the roots tangent to v with the following additional information. Let A be a real bunch such that $B = B(A)$ is a bar of $\mathbb{R}T_v(f)$. Then :

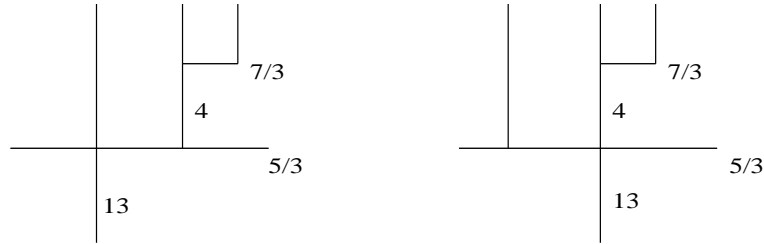
- draw the trunks on B realising the subbunches of A keeping the clockwise order of the roots (i.e. the order of the coefficients c_i in (6.1)),
- whenever B gives a new Puiseux pair to some roots of A (that can be easily computed from $T_+(f)$) we mark 0 on B and grow at it the unique sub-bunch of A with $c_i = 0$, i.e.

consisting of the roots that do not have the new Puiseux pair at B . Hence we are able to determine from the tree also the sub-bunches with positive and negative c_i . Graphically, we identify $0 \in B$ with the point of B that belongs to the trunk supporting B .

Example 6.6. Consider $f(x, y) = x(x^3 - y^5)(x^3 + y^5)$ and $g(x, y) = x(x^3 - y^5)(x^3 - 2y^5)$ of Example 1.1. The real trees $\mathbb{RT}_{(0,1)}(f)$, and $\mathbb{RT}_{(0,1)}(g)$, drawn below, are different.



Example 6.7. Consider $f(x, y) = x(x^3 - y^5)((x^3 - y^5)^3 - y^{17})$, $g(x, y) = x(x^3 + y^5)(x^3 - y^7)(x^6 + x^3y^5 + y^{10})$ as in remark 6.5. The real trees $\mathbb{RT}_{(0,1)}(f)$, and $\mathbb{RT}_{(0,1)}(g)$, drawn below, are different.



Definition 6.8. Let $f(x, y)$ be an analytic function germ. The *real tree model* $\mathbb{RT}(f)$ of f is defined as follow.

- Draw a bar B_0 that is identified with S^1 . We define $h(B_0) = 1$ and call B_0 the *ground bar*. We mark $m(B_0) := 2 \text{mult}_0 f(x, y)$ below the ground bar.
- Grow on B_0 non-trivial individual $\mathbb{RT}_v(f)$ for $v \in S^1$, keeping the clockwise order.
- Let v_1, v_2 be any two subsequent unit vectors for which $\mathbb{RT}_v(f)$ is nontrivial. Mark on B_0 of $\mathbb{RT}(f)$ the sign of f in the sector between v_1 and v_2 . Note that one such sign determines all the other signs between two subsequent unit vectors for which $\mathbb{RT}_v(f)$ is nontrivial (passing v changes this sign if and only if $\mathbb{RT}_v(f)$ contains an odd number of roots.)

If the leading homogeneous part f_m of f satisfies $f_m^{-1}(0) = 0$ then B_0 is the only bar of $\mathbb{RT}(f)$.

To each real bunch A of the real tree model $\mathbb{RT}(f)$ we associate a *generic demi-branch associated to A* , $\gamma_{A, \text{gen}} : x = \lambda_{A, \text{gen}}(y)$, where $\lambda_{A, \text{gen}}$ is given by

$$(6.2) \quad \lambda_{A, \text{gen}}(y) = \lambda_A(y) + cy^{h(A)} + \dots$$

with the constant c generic in \mathbb{R} . *The characteristic exponents* and respectively *the signs of characteristic coefficients associated to A* are those of $\gamma_{A,gen}$ that are $\leq h(A)$, respectively correspond to the exponents $< h(A)$.

Let B be a bar of $\mathbb{RT}(f)$ and let T_i be the trunks grown on B . Then, in the complex case, the multiplicity $m(B)$ of B equals the sum of the multiplicities of T_i . For the real tree model this is no longer true, $m(B) - \sum_i m(T_i)$ can be strictly positive, though it is always even. Note that in the real case each complex root is counted twice, once for $y \geq 0$ and once for $y \leq 0$.

Example 6.9. Let $f(x, y)$ be a singular germ weighted homogeneous with weights $w_1 > w_2$. Then $\mathbb{RT}(f)$ has exactly two trunks grown on B_0 corresponding to the y -axis, $y \leq 0$ and $y \geq 0$. Each of these trunks is bounded by a bar of height w_1/w_2 . These are all bars of the tree.

Remark 6.10. Let $B = B(A)$ be a bar of $\mathbb{RT}_v(f)$ such that $m, \beta_1, \beta_1/m < 2$, is the Puiseux characteristic sequence of $\gamma = \gamma_{A,gen} : x = \lambda_{A,gen}(y)$, and $h(A) = \beta_1/m$. Then B admits the opposite bar $-B$ of $\mathbb{RT}_{-v}(f)$ of the same height with the generic arc that is given in the system of coordinates $x' = -x, y' = -y$ by the same formula $x' = \lambda_{-A,gen}(y') = \lambda_{A,gen}(y')$ and $m(-B) = m(B)$. Moreover, let T_0 be the trunk that grows on B at 0, that is consisting of the roots that do not take a new Puiseux pair at $h(B)$. Then there is a trunk that grows on $-B$ at 0 that have the same multiplicity as T_0 .

Definition 6.11. We call two real trees $\mathbb{RT}(f), \mathbb{RT}(g)$ *isomorphic* if there is a homeomorphism φ of their ground bars sending one tree to the other and preserving the multiplicities and heights of bars and signs of the characteristic coefficients. If, moreover, φ preserves the orientation we call the trees *orientably isomorphic*.

Example 6.12. Let $1 < p < q$ be odd numbers. Then the real trees of $f(x, y) = x^p - y^q$ and $g(x, y) = x^p + y^q$ are not orientably isomorphic.

6.4. Effect of a blowing-up. Let $\pi : \widetilde{\mathbb{R}^2} \rightarrow \mathbb{R}^2$ be the blowing-up of the origin. Fix a point $\tilde{0} \in \widetilde{\mathbb{R}^2}$ on the exceptional divisor such that $f \circ \pi$ at $\tilde{0}$ is not the m -th power of the equation defining the exceptional divisor. Let v and $-v$ be the two opposite unit vectors of \mathbb{R}^2 corresponding to $\tilde{0}$. Then $\mathbb{RT}((f \circ \pi)_{\tilde{0}})$ is determined by $\mathbb{RT}_v(f)$ and $\mathbb{RT}_{-v}(f)$ and a choice of orientation of $(\widetilde{\mathbb{R}^2}, \tilde{0})$ in the following way.

Fix a bar $B = B(A)$ of $\mathbb{RT}_v(f)$ and $\mathbb{RT}_{-v}(f)$. Let $\gamma_{A,gen}$ denote its generic demi-branch, $h = h(A)$ its height, and $m(A)$ the multiplicity of A . By m, β_1, \dots we denote the characteristic sequence of $\gamma_{A,gen}$ (note that in general $m \neq m(A)$). B gives rise to a bar \tilde{B} or a couple of bars \tilde{B}_+, \tilde{B}_- of $\mathbb{RT}((f \circ \pi)_{\tilde{0}})$ that we describe by giving their generic demi-branches, heights, and multiplicities.

Denote by $\tilde{\gamma}$ the strict transform of $\gamma = \gamma_{A,gen}$. Then the Puiseux characteristic sequence of $\tilde{\gamma}$ is given by (4.3) and the sign of characteristic exponents are given by Proposition 4.3. The height of new bars can be computed using Proposition 4.5.

- (1) If $\beta_1/m > 2$ or $m = 1$ (γ smooth), then $h \geq 2$.
 If $h > 2$, then $\tilde{\gamma}$ and $h(\tilde{B}) = h - 1$ define a bar \tilde{B} , and $m(\tilde{B}) = m(B)$.
 If $h = 2$ then B gives the ground bar \tilde{B}_0 of $\mathbb{RT}((f \circ \pi)_{\tilde{0}})$.

- (2) The exceptional divisor is a root of $f \circ \pi$ of multiplicity $\text{mult}_0 f(x, y)$. It gives rise to two bars of $\mathbb{RT}((f \circ \pi)_{\tilde{0}})$ of infinite height.
- (3) Let $\beta_1/m < 2$ and $h = \beta_1/m$. Then B breaks down into two parts relative to the sign of c in (6.2), i.e. the sign of the first characteristic coefficient of γ . Denote these two different generic γ by γ_+ and γ_- respectively.

The strict transform $\tilde{\gamma}_+$ of γ_+ , and $h(\tilde{B}_+) = \frac{m}{\beta_1 - m}$ gives a new bar \tilde{B}_+ . One of the demi-branches of the exceptional divisor grows on \tilde{B}_+ and divides it into two parts. We note that $\tilde{\gamma}_+$ grows on one of these part. The other part corresponds to (a half of) $-B$, see Remark 6.10. Similarly the strict transform $\tilde{\gamma}_-$ of γ_- , and $h(\tilde{B}_-) = \frac{m}{\beta_1 - m}$ gives a bar \tilde{B}_- . The multiplicities of new bars are given by

$$m(\tilde{B}_+) = m(\tilde{B}_-) = m(B) - m(A') + \text{mult}_0 f(x, y),$$

where A' is the subbunch of A of the roots that does not take a new Puiseux pair at $h(B)$. The strict transforms of roots of A' do not grow neither on \tilde{B}_+ nor on \tilde{B}_- .

- (4) Let $\beta_1/m < 2$ and $h > \beta_1/m$ ($h < \beta_1/m$ cannot happen).

Then B grows over a bar B' with $h(B') = \beta_1/m$ and gives rise to a bar \tilde{B} that grows either on \tilde{B}'_+ or on \tilde{B}'_- , $h(\tilde{B}) = \frac{m}{\beta_1 - m}h - 1$ as follows from (5) of Proposition 4.5, and $m(\tilde{B}) = \frac{\beta_1 - m}{m}m(B)$.

Each bar of $\mathbb{RT}((f \circ \pi)_{\tilde{0}})$ comes from a bar of $\mathbb{RT}_v(f)$ or $\mathbb{RT}_{-v}(f)$. The only possible exception could be the ground bar \tilde{B}_0 of $\mathbb{RT}((f \circ \pi)_{\tilde{0}})$ if there is no bar of $\mathbb{RT}_v(f)$ or $\mathbb{RT}_{-v}(f)$ with $m(B) = 1$ and $h(B) = 2$.

The effect of a blowing-up on the tree can be also expressed in terms of horn, see subsection 7.3 below.

7. HORNS AND ROOT HORNS

In this section we characterise the real tree model of $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ in terms of the real analytic geometry without explicit referring to the complex Newton-Puiseux roots of f . This characterisation is used to show the blow-analytic invariance of the real tree model. The main idea is to replace a real bunch of roots by a geometric object, a horn.

7.1. Horns. Let $\gamma(t)$ be a reduced real analytic demi-branch given by (4.1). Define the *horn-neighbourhood of γ of exponent $\xi > 1$ and width $C > 0$* by

$$H_\xi(\gamma; C) = \{(x, y); \text{dist}((x, y), \text{image}(\gamma)) \leq C|(x, y)|^\xi\}.$$

By a *horn-neighbourhood of γ of exponent $\xi > 1$* we mean $H_\xi(\gamma; C)$ for C large and we denote it by $H_\xi(\gamma)$.

Remark 7.1. In order to simplify the exposition we use the following convention

- (1) If $O(\gamma_1, \gamma_2) \geq \xi$ then we say that $H_\xi(\gamma_1) = H_\xi(\gamma_2)$ by meaning that for any $C_1 > 0$ there is $C_2 > 0$ such that

$$H_\xi(\gamma_1; C_1) \subset H_\xi(\gamma_2; C_2), \quad H_\xi(\gamma_2; C_1) \subset H_\xi(\gamma_1; C_2)$$

(2) For similar reasons, if the demi-branch is represented by (4.1), we write

$$H_\xi(\gamma) = \{(x, y); |x - \lambda(y)| \leq C|y|^\xi\},$$

for C sufficiently large.

Let $H = H_\xi(\gamma)$ be a horn. Then by a *generic arc* γ_H in H we mean a demi-branch given by

$$(7.1) \quad x = \lambda_{H, \text{gen}}(y) = \lambda(y) + cy^\xi + \dots, \quad y \geq 0,$$

where $c \in \mathbb{R}$ is a generic constant. The *characteristic exponents* of H are those of γ_H that are $\leq \xi$. Similarly we define *the signs of characteristic coefficients* of H taking into account the exponents $< \xi$.

Given f and γ as above. Fix $\xi \geq 1$ and expand

$$(7.2) \quad f(\lambda(y) + zy^\xi, y) = P_{f, \gamma, \xi}(z)y^{\text{ord}_\gamma f(\xi)} + \dots,$$

where the dots denote higher order terms in y and $\text{ord}_\gamma f(\xi)$ is the smallest exponent with non-zero coefficient. This coefficient, $P_{f, \gamma, \xi}(z)$, is a polynomial function of z .

7.2. Root horns. Let A be a real bunch of $\mathbb{R}T_v(f)$. Then A defines a horn

$$H_A := \{(x, y); |x - \lambda_A(y)| \leq C|y|^{h(A)}\},$$

where C is a large constant. Then $\gamma_H = \gamma_{A, \text{gen}}$, see (6.2). A horn that equals H_A for a bunch A is called a *root horn*.

Proposition 7.2. *Let H be a horn of exponent ξ . Then H is a root horn for $f(x, y)$ if and only if $P_{f, \gamma_H, \xi}(z)$ has at least two distinct complex roots. If this is the case, $H = H_A$, then $h(A) = \xi$ and $m_A = \deg P_{f, \gamma_H, \xi}$.*

Proof. Suppose that $H = H_A$ and let $A = \{\gamma_1, \dots, \gamma_{m_A}\}$ be the corresponding bunch of roots. These roots are truncations of complex Newton-Puiseux roots of f :

$$(7.3) \quad \gamma_{\mathbb{C}, k} : x = \lambda_k(y) = \lambda_H(y) + a_{\xi, k}y^\xi + \dots, \quad 1 \leq k \leq m_A$$

with λ_H real and $a_{\xi, k} \in \mathbb{C}$. Denote by $\gamma_{\mathbb{C}, j} : x = \lambda_j(y)$, $j = m_A + 1, \dots, m$, the remaining complex Newton-Puiseux roots of f . Then

$$f(\lambda_H(y) + zy^\xi, y) = u(x, y) \prod_{i=1}^m (\lambda_H(y) - \lambda_i(y) + zy^\xi) = P_{f, \gamma_H, \xi}(z)y^{\text{ord}_{\gamma_H} f(\xi)} + \dots,$$

where $u(0, 0) \neq 0$. Note that $O(\lambda_H, \lambda_j) < \xi$ for $j > m_A$. Therefore

$$\begin{aligned} P_{f, \gamma_H, \xi_H}(z) &= u(0, 0) \prod_{i=1}^{m(A)} (z - a_{\xi, i}), \\ \text{ord}_{\gamma_H} f(\xi_H) &= m_A \xi_H + \sum_{j=m(A)+1}^m O(\lambda_H, \lambda_j). \end{aligned}$$

By construction of the tree there are at least two roots γ_i and γ_j of (7.3) such that $O(\lambda_i, \lambda_j) = \xi$. Thus $P_{f, \gamma_H, \xi_H}(z)$ has at least two distinct complex roots.

Let $H = H_\xi(\gamma)$ be a horn, where

$$\gamma : x = \lambda_H(y) + a_\xi y^\xi + \cdots.$$

By the Newton algorithm for computing the complex Newton-Puiseux roots of f to each root z_0 of $P_{f,\gamma,\xi}$ of multiplicity s correspond exactly s Newton-Puiseux roots of f , counted with multiplicities, of the form

$$\gamma_0 : x = \lambda_H(y) + (a_\xi + z_0)y^\xi + \cdots.$$

(This is essentially the way the Newton-Puiseux theorem is proved as in [32].) Thus, if $P_{f,\gamma,\xi}$ has at least two distinct roots, then there exist at least two such Newton-Puiseux roots with contact order equal to ξ . This shows that H is of the form H_B , as claimed. \square

7.3. Real construction of the real tree model. The root horns can be ordered by inclusion and by the clockwise order around the origin. Thus $V = H_A$ is contained in $V' = H_{A'}$ if and only if bunch A is contained in A' . The height $h(A)$ and the multiplicity $m(A)$ are given by Proposition 7.2

Let $\gamma_{V,gen}$ be a generic arc associated to the horn $V = H_A$. Let γ be any complex root in A . Then the Puiseux characteristic exponents of γ that are $< h(A)$ and the corresponding signs of characteristic coefficients are those of $\gamma_{V,gen}$. In particular, let \tilde{A} be a sub-bunch of A . Then $\gamma_{H_{\tilde{A}},gen}$ contains the information whether \tilde{A} takes a new Puiseux pair at $h(A)$ and, if this is the case, the sign of the characteristic coefficient at $h(A)$.

Thus the entire real tree model $\mathbb{RT}(f)$ can be obtained from the knowledge of the root-like horns and their numerical invariants.

Corollary 7.3. *Let $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ and $g : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ be real analytic function germs. If f and g are blow-analytically equivalent by an orientation preserving homeomorphism h then the real trees of f and g are orientably isomorphic.*

Proof. By section 3, f and g are cascade blow-analytically equivalent. By Theorem 5.1 and Corollary 5.3, the image by h of a root-like horn is a root-like horn with the same numerical invariants. \square

8. END OF PROOF OF THEOREM 0.1

We prove the theorem only in the oriented case. In (1) of Theorem 0.1 it means that the blow-analytic homeomorphism preserves the orientation. In (2) it means that the isomorphism $\Phi : M \rightarrow \tilde{M}$, cf. subsection 1.3, blows down to an orientation preserving homeomorphism $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$. In (3) it means that then the real trees of f and g are orientably isomorphic.

(1) \iff (2) by Proposition 1.3 and Section 3. (2) \implies (3) by Corollary 7.3. We show (3) \implies (2).

Let $\pi : \widetilde{\mathbb{R}^2} \rightarrow \mathbb{R}^2$ be the blowing-up of the origin. If both $f \circ \pi$ and $g \circ \pi$ are normal crossings then $\pi : \widetilde{\mathbb{R}^2} \rightarrow \mathbb{R}^2$ is the minimal resolution of f and of g and the claim follows. Otherwise, by Lemma 3.4, there is a one-to-one correspondence between the non-normal crossings points of $f \circ \pi$ and $g \circ \pi$. Fix such a pair of points $p_1 \in \pi^{-1}(0)$, $p_2 \in \pi^{-1}(0)$. By subsection 6.4 the real trees of $(f \circ \pi, p_1)$ and $(g \circ \pi, p_2)$, for a choice of local orientations,

coincide. Thus, by the inductive assumption, there are neighbourhoods U_1 and U_2 of p_1 and p_2 respectively, such that the minimal resolutions of $f \circ \pi|_{U_1}$ and $g \circ \pi|_{U_2}$ are isomorphic. This isomorphism induces an analytic isomorphism of punctured neighbourhoods $U_1 \setminus p_1$ and $U_2 \setminus p_2$.

For a pair of corresponding points p_1 and p_2 , fix a half-line l_1 in \mathbb{R}^2 at 0 representing p_1 . Let l_2 be the half-line corresponding to l_1 by the identification of both trees. A choice of half-line l_1 induces on (\mathbb{R}^2, p_1) an orientation, the one that lifts the canonical orientation of \mathbb{R}^2 at the points of l_1 . Note that replacing l_1 and l_2 by the opposite half-lines reverses both orientations, and therefore gives the same isomorphism of minimal resolutions of $(f \circ \pi, p_1)$ and $(g \circ \pi, p_2)$. Thus the identification of both trees gives a coherent system of orientations and the local isomorphisms of minimal resolutions glue together, see Remark 1.4. \square

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